Kink solitons in quadratic-cubic nonlinear dispersive media

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We show analytically that the coexistence of quadratic and cubic nonlinearities in dispersive media offers kink solitons with the Fermi-Dirac distribution. The underlying principle and the ubiquity of the present solitons are discussed.

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I. INTRODUCTION

The soliton phenomenology has now become ubiquitous in modern sciences, and can be found in many branches of physics [1]. Of these, along with the Korteweg-de Vries and the sine-Gordon systems, the type that can be described by a nonlinear Schrödinger-type equation will be representative of solitons. A well known example of this type of soliton is found in the laser beam (or pulse) propagation in a Kerr-type nonlinear medium that includes manmade structures such as fibers and waveguides [2]. More recently, signals that show evidence for the microwave soliton have been detected [3] by using intensified magnetostatic-wave propagation in a ferromagnetic thin film. As is well known, there exist two kinds of solitons in the canonical $(1+1)$-dimensional cubic nonlinear Schrödinger equation (NLSE): bright and dark solitons. In addition to these two, the third type, termed a kink soliton, was found to exist in a variety of physical contexts. Of several models that can support a kink soliton or a kinky solitary wave, the most representative is that predicted by the sine-Gordon equation [4]. Aside from the sine-Gordon kinks, in the framework of continuum approximation for nonlinear monatomic chains, Flytzanis, Pnevmatikos, and Remoissenet [5] presented a kink soliton solution of a generalized Boussinesq equation that they derived. In the context of nonlinear optics, shock-type solitary-wave solutions with tachonic properties were obtained by Christodoulides [6] for the case of nonlinear interaction of two optical waves in a Raman medium. Through a nonlinear medium with strong dissipation, a Taylor shock wave could propagate as a kink soliton. Concerning propagation of electromagnetic transients in a quadratic nonlinear dispersive medium, Xu, Auston, and Hasegawa [7] recently predicted the Taylor shock-wave solution in a strongly dissipative regime. More recently, in the nonlinear optics context, Agrawal and Headley III [8] showed that a $(1+1)$-dimensional generalized NLSE that includes the effect of intrapulse stimulated Raman scattering offers a kink soliton, which represents a shock front that propagates undistorted inside a dispersive nonlinear medium. Subsequently, as a higher-dimensional extension of self-consistent kink formation, a bright-kink symbion resulting from the combined effect of self-trapping and intrapulse stimulated Raman scattering was predicted by Hayata and Koshiba [9]. We present in this paper a kink soliton with the Fermi-Dirac distribution. This type of kink soliton arises from the coexistence of quadratic and cubic nonlinearities in a dispersive medium. The underlying principle of the present soliton is described, and therein two different types (type I and type II) are found to exist. Emphasis is on its ubiquity in various physical models that include, e.g., mathematical ecology, chemical kinetics, radiation-matter interactions, and nonlinear optics.

II. FERMI-DIRAC KINK SOLITARY WAVES

First we consider a generic version of a quadratic-cubic nonlinear Schrödinger-type equation

$$\kappa u_t = u_{xx} + c_1 u + c_2 u^2 + c_3 u^3 + c_0 , \tag{1}$$

where u is the field amplitude that depends on x and t $[u=u(x,t)]$, c_j $(j=0-3)$ is a real constant corresponding to the coefficient of u^j (assume $c_2 c_3 \neq 0$), and κ is a complex parameter. For instance, for reaction-diffusion systems [10], $\kappa \equiv 1$, while for paraxial wave propagation, $\kappa \equiv \pm i$.

Through a heuristic manner we have found that Eq. (1) admits of a stationary (a static) solitary-wave solution $(u_t \equiv 0)$ [11]

$$u(x) = u_R + u_L f_{\rm FD}(\alpha x) , \tag{2a}$$

with

$$f_{\rm FD}(\xi) \equiv [\exp(\xi)+1]^{-1} , \tag{2b}$$

provided that

$$c_0 = \alpha^2 (u_R + 3u_R^2/u_L + 2u_R^3/u_L^2) , \tag{3a}$$

$$c_1 = -\alpha^2 (1 + 6u_R/u_L + 6u_R^2/u_L^2) , \tag{3b}$$

$$c_2 = \alpha^2 (3/u_L + 6u_R/u_L^2) , \tag{3c}$$

$$c_3 = -2\alpha^2/u_L^2 . \tag{3d}$$

Here $u_L \equiv u(-\infty) \neq 0$, $u_R \equiv u(\infty)$, and α is a positive constant. It is interesting to note that Eq. (2b) is identical to the Fermi-Dirac distribution that is familiar in solid-state physics.

As will be discussed below, when the driving term of Eq. (1) vanishes (i.e., $c_0 \equiv 0$), one will find several important physical systems that can be modeled with it. For

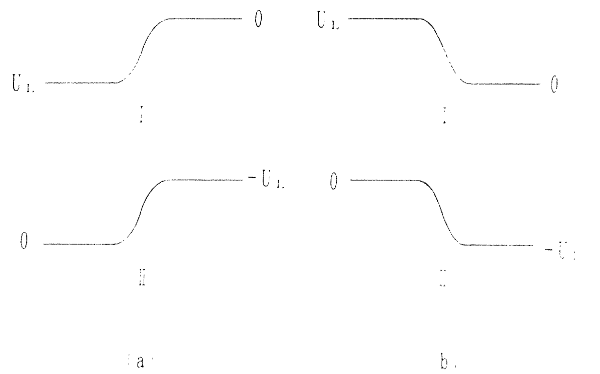

FIG. 1. Schematic illustrations of (a) kink ($u_L<0$) and (b) antikink ($u_L>0$) solitons in a quadratic-cubic nonlinear dispersive medium. The definitions of the type-I and the type-II kinks are given in the text.

this reason, we shall concentrate in what follows on the homogeneous (the source-free) version of Eq. (1). Setting $c_0=0$ on Eq. (3a), among the three cases we find only the following two cases nontrivial:

$$u_R=0 \quad \text{or} \quad u_R=-u_L \ . \tag{4}$$

Here the solution, $u_R=-u_L/2$, has been rejected because, as obvious from Eq. (3c), the quadratic nonlinearity vanishes ($c_2=0$) [12]. With Eq. (4) being substituted, Eq. (2a) becomes

$$u(x)=u_L f_{\rm FD}(\alpha x) \quad \text{for } u_R=0 \text{ (type I)} \ , \tag{5a}$$

$$u(x)=u_L[f_{\rm FD}(\alpha x)-1] \quad \text{for } u_R=-u_L \text{ (type II)} \ , \tag{5b}$$

where (c_1, c_2) in Eqs. (3b) and (3c) are reduced to $(c_1, c_2)=(-\alpha^2, 3\alpha^2/u_L)$ for $u_R=0$ (type I), and $(c_1, c_2)=(-\alpha^2, -3\alpha^2/u_L)$ for $u_R=-u_L$ (type II). For later convenience we identify the kink solution of Eq. (5a) [Eq. (5b)] with the type-I (type-II) kink. Schematic illustrations of the kink profiles are shown in Fig. 1. According to the terminology used for the sine-Gordon context, the negative u_L ($u_L<0$) yields a kink solution ($u_L<u_R$), whereas the positive u_L ($u_L>0$) yields an antikink solution ($u_L>u_R$). Apparently, α^{-1} features the steepness of the kinky section. To evidence the solitonic feature of the present kink solitary wave, numerical simulations have been performed with a computational tool previously developed by us [13]. Example results are shown in Fig. 2. In both types the solitary wave being input at $t=0$ is stable and remains unchanged even after the propagation over ten soliton units.

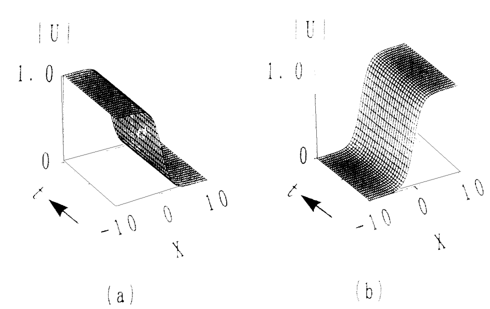

FIG. 2. Evolution of (a) type-I and (b) type-II solitary wave along the t axis. The parameters in Eq. (1) are set to be $\kappa=-i$, $c_0=0$, $c_1=-1$, $c_2=3$ (-3) for the type-I (II) kink, and $c_3=-2$. Those in Eq. (5) are $u_L=\alpha=1$. The kink solution of Eq. (5) is launched at $t=0$. The total propagation time attains 5π, which coincides with ten soliton units, where one soliton unit is taken to be $\pi/2$. Note that for $\kappa=1$ with other parameters unchanged, the exact same results have been obtained.

III. SPECIFIC EXAMPLES IN PHYSICAL SYSTEMS

Below, in diverse branches of physical systems we shall find relevance to the present kink solutions. To our knowledge, the simplest example that can be modeled in terms of the quadratic-cubic nonlinear Schrödinger-type equation will be seen in the Fisher-type nonlinear diffusion equation for ecological wave evolutions [14]

$$\begin{aligned} u_t &= u_{xx}-u(a-u)(1-u) \\ &= u_{xx}-au+(a+1)u^2-u^3 \ , \end{aligned} \tag{6}$$

where u represents the local density of a particular species, and a is a phenomenological real parameter $(-1\le a<1)$. Through brief algebra we have found that with $u_t\equiv 0$, Eq. (6) allows the type-I kink solution

$$u(x)=f_{\rm FD}(x/2^{1/2}) \quad \text{for } a=\tfrac{1}{2} \ . \tag{7}$$

Note that for any value of a, we have proved that Eq. (6) does not include the type-II solution.

Subsequently we consider the reaction-diffusion equations based on a piecewise linearized FitzHugh-Nagumo model [15,16]

$$u_t=u_{xx}-\varepsilon^{-1}u(u-1)[u-(v+b)/a] \ , \tag{8a}$$

$$v_t=u-v \ , \tag{8b}$$

where the model is assumed to consist of two chemical species, u and v, that evolve in the space-time and a, b, and ε are real parameters [16]. For instance, for a continuous excitable medium the key parameters are b and ε. The former determines the excitation threshold, and the latter provides the relationship of the time scales of the fast (activator u) and the slow (inhibitor v) variable. Since in usual cases the time scale of u is much faster than that of v, the parameter ε is typically smaller than unity. Note that the excitability of a system may be defined by ε^{-1}. When $v_t\equiv 0$, Eqs. (8) are joined to the single equation

$$\begin{aligned} u_t &= u_{xx}-\varepsilon^{-1}u(u-1)[u-(u+b)/a] \ , \\ &= u_{xx}-b(\varepsilon a)^{-1}u-\varepsilon^{-1}[(1-b)a^{-1}-1]u^2 \\ &\quad -\varepsilon^{-1}(1-a^{-1})u^3 \ . \end{aligned} \tag{9}$$

Through algebra with $u_t\equiv 0$, we have found that in contrast to the aforementioned Fisher-type system, Eq. (9)

offers both the type-I [Eq. (5a)] and the type-II [Eq. (5b)] kinks with

$$u_L=\mu\ ,\quad \alpha=[(a-1)(2\varepsilon a)^{-1}]^{1/2}\quad \text{for } a=2b+1\ , \tag{10a}$$

$$u_L=2\mu\ ,\quad \alpha=[2(a-1)(\varepsilon a)^{-1}]^{1/2}\quad \text{for } a=b/2+1\ , \tag{10b}$$

where $\mu=1$ (-1) for the type-I (II) kink. Note that with $\mu=1$ and $a=(1-\varepsilon)^{-1}$ in Eq. (10a), the solution becomes identical to Eq. (7).

The possibility of supporting the kink wave could also be found in electromagnetics. In the context of radiation-matter interactions, stationary propagation of far-infrared electromagnetic polaritons in a nonlinear dispersive crystal medium far from the material resonance may be governed by an extended Boussinesq-type equation [7,17]

$$u_{xxxx}+(\gamma^2-1)u_{xx}+6(u^2)_{xx}+\theta(u^3)_{xx}=0\ , \tag{11a}$$

with

$$\theta=-36\varepsilon_{\rm dc}\chi^{(3)}/(\chi^{(2)})^2\ , \tag{11b}$$

where u is a principal component of the electric field, x is a scaled local time in the frame of reference moving with the signal velocity, γ is a real parameter that represents the reciprocal signal velocity, $\varepsilon_{\rm dc}$ is the static dielectric constant, and $\chi^{(2)}$ $(\chi^{(3)})$ represents the quadratic (cubic) nonlinearity. With vanishing θ, Eq. (11a) coincides with the equation derived by Xu, Auston, and Hasegawa [7] for a quadratic nonlinear dispersive medium. In the context of the continuum limit of nonlinear monatomic chains, Flytzanis, Pnevmatikos, and Remoissenet [5] derived a generalized Boussinesq equation that is similar to Eq. (11a). For the equation they presented a kink soliton solution. Although the solution they derived may include the present Fermi-Dirac kink, they did not mention the relevance to it.

To derive the kink solution of Eq. (11a), first we set $u_t\equiv 0$ in Eq. (1), and differentiate it with x:

$$u_{xxxx}+c_1u_{xx}+c_2(u^2)_{xx}+c_3(u^3)_{xx}=0\ . \tag{12}$$

Comparison of Eqs. (11a) and (12) leads straightforwardly to

$$u_L=\pm 2\theta^{-1}[12-\theta(\gamma^2-1)]^{1/2}\ , \tag{13a}$$

$$u_R=-\theta^{-1}\{2\pm[12-\theta(\gamma^2-1)]^{1/2}\}\ , \tag{13b}$$

$$\alpha=(-\theta/2)^{1/2}|u_L|=\{-2\theta^{-1}[12-\theta(\gamma^2-1)]\}^{1/2}\ . \tag{13c}$$

From Eq. (13c), θ must be negative $(\theta<0)$, which indicates that only the self-focusing type [$\chi^{(3)}>0$ in Eq. (11b)] is allowed. Because u must be real, from Eqs. (13a) and (13b) it must be required that

$$\gamma^2>12/\theta+1\ . \tag{14}$$

For $-12<\theta<0$, Eq. (14) holds unconditionally, whereas for $\theta\le -12$, this requirement is met solely for $|\gamma|>(12/\theta+1)^{1/2}$. When $u_R=0$ (type-I kink), Eqs. (13) are reduced in the form

$$u_L=-4/\theta\ ,\quad \alpha=(-8/\theta)^{1/2}\ , \tag{15}$$

with

$$|\gamma|=(1+8/\theta)^{1/2}\ , \tag{16}$$

where $\theta<-8$. It should be stressed here that because γ governs the reciprocal velocity normalized by the speed of light in the material medium, $|\gamma|<\varepsilon_{\rm dc}^{1/2}$ permits of superluminal (tachonic) propagation of the pulse, which means that the kink velocity does exceed the speed of light in vacuum. This possibility is extremely interesting in the sense that information carried by the kink could be transmitted with a superluminal velocity.

Finally we shall explore whether the present kink field could in principle be generated in the optical regime of electromagnetic spectra. To answer this question, as an example we consider a phase-matched collinear interaction of a fundamental (ω) and a second-harmonic (2ω) laser beam in a quadratic-cubic nonlinear medium. For slowly varying envelopes along the propagation (the t) axis, the basic equations for the parametrically coupled optical envelopes, $u(\omega)$ and $v(2\omega)$, can be written as [18]

$$-i2Bu_t=u_{xx}+(\varepsilon_1-B^2)u+2d_1u^*v+a_1(|u|^2+2|v|^2)u\ , \tag{17a}$$

$$-i4Bv_t=v_{xx}+4(\varepsilon_2-B^2)v+4d_2u^2+4a_2(|v|^2+2|u|^2)v\ , \tag{17b}$$

where B is the phase constant along the t axis, ε is the relative permittivity, $d(a)$ accounts for contribution from the quadratic (cubic) nonlinearity, the subscript j $(j=1,2)$ indicates the quantity of the $j\omega$ component, and the asterisk denotes complex conjugate. Unlike the three examples discussed above, in the present context the field variables u and v are in general complex. We have found that when $u_t=v_t\equiv 0$, Eqs. (17) can yield both the type-I and the type-II kink solutions:

$$u(x)=u_Lf_{\rm FD}(\alpha x)\ ,\quad v(x)=v_Lf_{\rm FD}(\alpha x)\quad \text{for type-I kink}\ , \tag{18a}$$

$$u(x)=u_L[f_{\rm FD}(\alpha x)-1]\ ,\quad v(x)=v_L[f_{\rm FD}(\alpha x)-1]\quad \text{for type-II kink}\ , \tag{18b}$$

with

$$B=(\varepsilon_2+\Delta\varepsilon/3)^{1/2}\ ,\quad \Delta\varepsilon\equiv\varepsilon_2-\varepsilon_1\ , \tag{19a}$$

$$\alpha=2(\Delta\varepsilon/3)^{1/2}\ , \tag{19b}$$

$$|u_L|=[2/(d_1d_2)]^{1/2}\Delta\varepsilon\ , \tag{19c}$$

$$v_L=\eta u_L\ , \tag{19d}$$

$$|\eta|=(2K_q)^{1/2}\ ,\quad K_q\equiv d_2/d_1\ , \tag{19e}$$

for

$$0.125 < K_c < 0.5 , \quad K_c \equiv a_2/a_1 , \tag{20a}$$

$$a_1 = -4d_1d_2/[3\Delta\varepsilon(1+4K_q)] , \tag{20b}$$

$$a_2 = -d_1d_2/[6\Delta\varepsilon(1+K_q)] . \tag{20c}$$

From Eqs. (20) the quadratic and the cubic nonlinearities should be interrelated through the relation

$$K_q = (1-8K_c)/(8K_c-4) . \tag{21}$$

For instance, for $K_q \sim 1$ ($d_1 \sim d_2 \equiv d$), Eqs. (20) and (21) become

$$a_1 \sim -4d^2/(15\Delta\varepsilon) , \quad a_2 \sim -d^2/(12\Delta\varepsilon) , \quad K_c \sim 0.3125 . \tag{22}$$

Finally we would like to mention the stability of the Fermi-Dirac kink against perturbations. It should be noted that there is a qualitative difference between topological kinks of the sine-Gordon model [4] and nontopological kinks as reported in Refs. [5–8]. Obviously the Fermi-Dirac kink that is presented in this paper belongs to the latter. According to the recent work by Kivshar and co-workers [19], the latter class of kinks may become unstable if the background becomes unstable.

IV. CONCLUSIONS

We have presented a kink soliton with the Fermi-Dirac profile, which arises from the coexisting quadratic and cubic nonlinearities in a dispersive medium. The underlying principle of the present soliton have been elucidated and its ubiquitous aspects in modern physical sciences have been confirmed by presenting typical example models in various physical contexts.

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$$u(x) = 2a[(b^2-4ac)^{1/2}\cosh(a^{1/2}x)+b]^{-1} ,$$

where $a \equiv -c_1 > 0$, $b \equiv 2c_2/3 > 0$, and $c \equiv -c_3/2$. Note that $u \to 0$ as $|x| \to \infty$. In the limit of $c_2 \to 0$, this coincides with the hyperbolic-secant-type bright soliton of a cubic NLSE.

[12] For $u_R/u_L = -\frac{1}{2}$, $u(x) = u_R \tanh(\alpha x/2)$. This is just a dark soliton of a cubic NLSE, which implies that the present Fermi-Dirac-type kink soliton includes the dark (the black) soliton as a special case.

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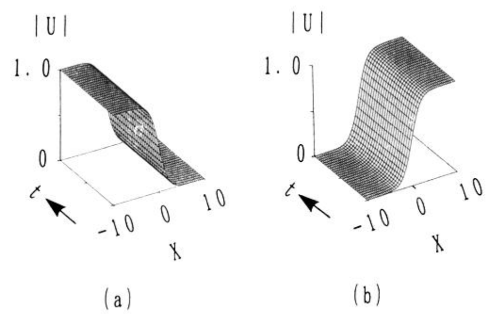

FIG. 2. Evolution of (a) type-I and (b) type-II solitary wave along the t axis. The parameters in Eq. (1) are set to be $\kappa=-i$, $c_0=0$, $c_1=-1$, $c_2=3$ (-3) for the type-I (II) kink, and $c_3=-2$. Those in Eq. (5) are $u_L=\alpha=1$. The kink solution of Eq. (5) is launched at $t=0$. The total propagation time attains 5π, which coincides with ten soliton units, where one soliton unit is taken to be $\pi/2$. Note that for $\kappa=1$ with other parameters unchanged, the exact same results have been obtained.